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1992 J. Phys. A: Math. Gen. 25 2065

(http://iopscience.iop.org/0305-4470/25/7/041)

J. Phys. A: Math. Gen. **25** (1992) 2065–2070. Printed in the UK

Level shift under the influence of relativistic point interaction potentials

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Received 7 June 1991

Abstract. Level shift of Dirac particles under the influence of point interaction potentials has been determined. A Green function method is used to obtain closed expressions for the level shift and perturbed wavefunctions, provided that the corresponding eigenvalue problem without point interaction potentials can be solved exactly. Several applications are discussed in detail.

1. Introduction

Point interaction potentials (PIPs) in one space dimension are widely used to approximate more structured and more complex short-ranged potentials. The term PIP refers to any arbitrary sharply peaked potential approaching the δ-function limit (a more rigorous definition can be found in the monograph of Albeverio *et al* 1988). Although solutions of the Schrödinger equation for PIPs are well defined, there exist some ambiguities regarding the definition of such potentials for the Dirac equation in one space dimension (Fairbairn *et al* 1973, Sutherland and Mattis 1981). The origin of this arbitrariness has been clarified recently (McKellar and Stephenson 1987, Calkin *et al* 1988), and it is related to the fact that the Dirac equation is linear in momentum rather than quadratic. In addition, Coutinho and Nogami (1990) have proved that PIPs cannot be defined for the Dirac equation in two and three space dimensions.

In recent papers, bound and scattering states and confining properties of isolated PIPs have been studied in detail (Domínguez-Adame and Maciá 1989a, Maciá and Domínguez-Adame 1991). We are naturally interested in the generalization of previous works to include the possible existence of non-singular potentials. This is not a trivial task since we must overcome the ambiguities concerning the relativistic PIP mentioned above. The aim of this work is to find solutions of the Dirac equation for interactions involving PIPs, whenever the corresponding eigenvalue problem without PIPs is solved exactly. A Green function method is used to compute the effects of PIPs on the unperturbed eigenvalues. We will show that there exists an exact (and in general) transcendental equation which determines energy levels in a unique way.

2. The effects of point interaction potentials

The Dirac equation in one space dimension for steady states can be written as ($\hbar = c = 1$)

$$\left[-\mathrm{i}\alpha\frac{\mathrm{d}}{\mathrm{d}x}+\beta m - E + U(x)+(v+\beta s)F(x)\right]\psi(x)=0 \tag{1}$$

0305-4470/92/072065+06$04.50

where α and β are 2×2 Hermitian, traceless matrices with square unity such that $\alpha\beta+\beta\alpha=0$, acting upon the two-component wavefunction ψ. Here $U(x)$ is any non-singular potential for which the Dirac equation can be solved; v and s denote the vector (the time component of a Lorentz vector) and scalar coupling constants, respectively. $F(x)$ is a positive, sharply peaked function at x_0, satisfying the limiting condition

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} F(x)\,\mathrm{d}x = 1$$

ε being a small positive parameter. By means of a space translation we can set $x_0=0$ without losing generality. The appropriate boundary condition around the point where the PIP is located reads (McKellar and Stephenson 1987)

$$\psi(0^+) = \mathrm{e}^{-\mathrm{i}\alpha(v+\beta s)}\psi(0^-) \tag{2}$$

which becomes independent of how the δ-function limit is taken.

Solutions of (1) may be written as

$$\psi(x) = -\int_{-\infty}^{\infty} \mathrm{d}x'\, G(x, x'; E)(v+\beta s)F(x')\psi(x') \tag{3}$$

where the Green function for the unperturbed problem is a 2×2 matrix satisfying the inhomogeneous differential equation

$$\left[-\mathrm{i}\alpha\frac{\partial}{\partial x}+\beta m - E + U(x)\right]G(x, x'; E) = I_2\delta(x-x') \tag{4}$$

subject to suitable boundary conditions. I_2 stands for the 2×2 unity matrix. Instead of solving (4), the Green function can also be obtained by explicit summation over unperturbed eigenstates, which are solutions of (1) for $v=s=0$. The relativistic Green function exhibits a jump discontinuity on the line $x=x'$, in contrast to the non-relativistic case. The value of the jump is obtained by integration of (4) in the vicinity of this line. The result is

$$G(x^+, x; E) - G(x^-, x; E) = \mathrm{i}\alpha. \tag{5}$$

The integral appearing in (3) is not well defined as $F(x)\to\delta(x)$ at the outset. The reason comes from the fact that $\psi(x)$ shows a discontinuity at $x=0$ to account for the singularity of the potential (recall that the Dirac equation is linear in the spatial derivative), and the product $\theta(x)\delta(x)$ is ill defined in a strict distribution theory sense, $\theta(x)$ being the Heaviside step function, as pointed out by McKellar and Stephenson (1987). The situation is even worse considering the limit $x\to0$ on both sides of (2), because $G(x, x'; E)$ is also discontinuous at $x=x'=0$. To overcome these difficulties, we solve the integral equation (3) for any sharply peaked function $F(x)$, and then take the δ-function limit.

According to the definition of PIPs, $F(x)$ vanishes for $|x|>\varepsilon$. Therefore, using the Dirac equation (1), one finds that the integral equation (3) is written, after integration by parts, as follows:

$$\begin{aligned}\psi(x) = -\mathrm{i}[G(x, x'; E)\alpha\psi(x')]_{-\varepsilon}^{\varepsilon} &+ \int_{-\varepsilon}^{\varepsilon} \mathrm{d}x'\, G(x, x'; E)[\beta m - E + U(x')]\psi(x') \\ &+ \mathrm{i}\int_{-\varepsilon}^{\varepsilon} \mathrm{d}x'\left(\frac{\partial}{\partial x'}G(x, x'; E)\right)\alpha\psi(x').\end{aligned} \tag{6}$$

The second term of the RHS vanishes in the limit $\varepsilon \to 0$, provided that $U(x)$ is not singular. The third term also vanishes for $|x| > \varepsilon$ as $\varepsilon \to 0$ because $G(x, x'; E)$ is a continuous function for $x \neq x'$, so that its derivative is non-singular outside the line $x = x'$. Hence the solution of (1) is found to be

$$\psi(x) = -iG(x, 0; E)\alpha[\psi(0^+) - \psi(0^-)] \qquad |x| > 0 \tag{7}$$

where $\psi(0^+)$ and $\psi(0^-)$ are related through (2). Therefore, we have obtained a closed form for the perturbed wavefunction. Notice that this result becomes independent of the exact shape of the function $F(x)$, so that arbitrariness in defining relativistic PIPs is avoided.

Taking the limits $x \to 0^-$ and $x \to 0^+$ in (7) and making use of (2) we obtain two consistency equations, namely

$$\det[1 + iG(0^-, 0; E)\alpha(e^{-i\alpha(v+\beta s)} - 1)] = 0 \tag{8a}$$

and

$$\det[1 - iG(0^+, 0; E)\alpha(e^{i\alpha(v+\beta s)} - 1)] = 0. \tag{8b}$$

Actually both conditions are just the same because of equation (5), as can easily be shown. Equation (8*a*) or (8*b*) determines the energy levels of the Dirac particle in a closed form, as long as the Green function for the unperturbed potential is known. Once the energy levels have been determined, the corresponding wavefunction is found by substitution of the appropriate value of E into (7). It should be emphasized that equations (7) and (8) are independent of the particular representation of the matrices α and β.

3. Results

To illustrate the method introduced above, we study three examples corresponding to different choices of the potential $U(x)$ in (1). In order to solve (8), we must specify a particular representation for the Dirac matrices. We set $\alpha = \sigma_x$ and $\beta = \sigma_z$, the σ being the 2×2 Pauli matrices.

3.1. Bound states of the isolated PIP

Setting $U(x) = 0$ we can use our method to find the bound energy levels of a Dirac particle interacting with an isolated PIP. The Green function for a free Dirac particle becomes ($E^2 < m^2$)

$$G(x, x'; E) = \frac{1}{2q} e^{-q|x-x'|}(i\sigma_x q \operatorname{sgn}(x - x') + \sigma_z m + E) \tag{9}$$

where $q = (m^2 - E^2)^{1/2}$ is real for bound states. Using either (8*a*) or (8*b*) we obtain

$$-q = (Ev + ms) \tan(v^2 - s^2)^{1/2}/(v^2 - s^2)^{1/2}. \tag{10}$$

This result agrees with that found by Domínguez-Adame and Maciá (1989a) in dealing with mixed (vector plus scalar) PIPs.

3.2. Relativistic Kronig–Penney model

The dynamics of Dirac electrons in a periodic array of δ-function potentials has been studied by several authors (Domínguez-Adame 1987, and references therein). Now we proceed to apply our method to this problem. The Dirac equation for such electrons is obtained from (1) setting $U(x)=0$ and replacing

$$F(x) \to \sum_{n=-\infty}^{\infty} F(x-nL) \tag{11}$$

L being the lattice period. The Bloch theorem ensures that the wavefunction is of the form

$$\psi(x) = -\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}x'\, G(x, x'+nL; E)(v+\beta s)F(x')\, \mathrm{e}^{\mathrm{i}knL}\, \psi(x') \tag{12}$$

where k is the crystal momentum. Equation (12) for a periodic arrangement of PIPs is the analogue of (3) for a single PIP. Therefore, we can apply equations (8) but replacing

$$G(0^{\pm}, 0; E) \to \sum_{n=-\infty}^{\infty} G(0^{\pm}, nL; E)\, \mathrm{e}^{\mathrm{i}knL}. \tag{13}$$

Without loss of generality we consider $E^2 > m^2$. Hence the free-particle Green function becomes

$$G(x, x'; E) = \frac{\mathrm{i}}{2\eta}\, \mathrm{e}^{\mathrm{i}\eta|x-x'|}(\sigma_x \eta\, \mathrm{sgn}(x-x') + \sigma_z m + E) \tag{14}$$

where $\eta = (E^2-m^2)^{1/2}$. After performing the summation of (13), provided that η has a small positive imaginary part to ensure convergence, one gets from (8) the following dispersion relation:

$$\cos kL = \cos(v^2-s^2)^{1/2} \cos \eta L + (Ev+ms)\frac{\sin(v^2-s^2)^{1/2}}{(v^2-s^2)^{1/2}}\frac{\sin \eta L}{\eta}. \tag{15}$$

In the particular case of pure vector PIPs ($v \neq 0$, $s=0$), equation (15) reduces to the dispersion relation obtained by Domínguez-Adame (1989) using the transfer matrix method.

3.3. Singular Dirac oscillator

The spectroscopy of singular harmonic oscillators are of interest in quark physics at small distances (Avakian *et al* 1987). Domínguez-Adame and Maciá (1989b) have studied the effects of scalar PIPs on the energy levels of the relativistic Ravndal oscillator (Ravndal 1982). In a recent paper, Moshinsky and Szczepaniak (1989) have considered an interesting interaction, in an attempt to describe a relativistic Dirac oscillator by means of an equation linear in both coordinates and momenta. A 1D version of the Dirac oscillator is obtained replacing m by $m+\mathrm{i}m\omega\alpha x$ in the free-particle Hamiltonian, ω being the frequency (Domínguez-Adame and González 1990). To study the spectroscopy of the singular Dirac oscillator we choose $U(x) = \mathrm{i}m\omega\beta\alpha x$ in equation (1). The matrix elements of the Green function

$$G(x, x'; E) = \begin{pmatrix} G_{++}(x, x'; E) & G_{+-}(x, x'; E) \\ G_{-+}(x, x'; E) & G_{--}(x, x'; E) \end{pmatrix} \tag{16}$$

for the 1D Dirac oscillator are given in the standard representation by (Domínguez-Adame 1991)

$$G_{\pm\pm}(\xi,\xi';E)=\frac{E\pm m}{\sqrt{8m\omega}}\frac{D(\lambda_\pm,-\xi_<)D(\lambda_\pm,\xi_>)}{D(\lambda_\pm,0)D(\lambda_\pm+1,0)} \tag{17a}$$

$$G_{\pm\mp}(\xi,\xi';E)=-\mathrm{i}\frac{(2m\omega)^{1/2}}{E\pm m}\left(\frac{\partial}{\partial\xi}\mp\frac{\xi}{2}\right)G_{\pm\pm}(\xi,\xi';E) \tag{17b}$$

where $\xi=(2m\omega)^{1/2}x$, $\lambda_\pm=(E^2-m^2)/2m\omega\pm\frac{1}{2}-\frac{1}{2}$, and $\xi_>(\xi_<)$ is the largest (smallest) value of (ξ,ξ'). Here $D(\lambda,z)$ denotes the parabolic cylinder functions. Using equations (8) and the well known properties of $D(\lambda,z)$ we obtain after a little algebra

$$-\tan(v^2-s^2)^{1/2}/(v^2-s^2)^{1/2}=[(\chi-\chi^{-1})v+(\chi+\chi^{-1})s]^{-1} \tag{18}$$

where

$$\chi=\chi(E)=\frac{E+m}{2\sqrt{m\omega}}\frac{\Gamma(-(E^2-m^2)/4m\omega)}{\Gamma(\frac{1}{2}-(E^2-m^2)/4m\omega)}. \tag{19}$$

This transcendental equation determines the energy levels of the singular Dirac oscillator, and has to be solved numerically. Unlike the non-relativistic case (Avakian *et al* 1987), the presence of PIPs changes not only the even harmonic-oscillator levels but also the odd ones.

4. Conclusions

We have proved that the effects of relativistic point interaction potentials can be exactly evaluated using a Green function technique. Our method becomes independent of how the δ-function limit is taken, so ambiguities in defining such potentials are overcome. Although we have performed a 1D treatment, this method can be extended to compute the effects of relativistic spherically symmetric δ-function potentials (Dittrich *et al* 1989, Domínguez-Adame 1990) on energy levels of 3D potentials. In particular, this realization will provide an alternative way of studying the eigenvalues of the Coulomb potential with relativistic spherically symmetric δ-function potentials (Dittrich *et al* 1991).

Acknowledgments

The authors wish to thank E Maciá for helpful comments and Dr J Dittrich for sending his paper prior to publication.

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